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Minimum Control Authority Plot: A Tool for Designing Thruster Systems

Peter J. Wiktor*
Engineering Arts, Seattle, Washington 98115

A new performance measure for thruster systems called the minimum control authority is defined. It is one of the most important performance measures since it corresponds to the worst case output of the thruster system. Various thruster system designs can easily be evaluated based on this performance measure. Several techniques are derived to calculate the minimum control authority as a function of the thruster configuration, the way in which the thrusters are controlled, and the physical mechanism that limits the thrusters' outputs. A technique to generate a plot of the worst case moment vs force-generating capability of a thruster system is also derived. This so-called minimum control authority plot can be used to evaluate the worst case output of various thruster system designs against each other and against the worst case disturbance forces they have to overcome. To illustrate the concept and demonstrate its usefulness, the minimum control authority plot is applied to evaluate various thruster system designs for the Gravity Probe B (GP-B) spacecraft. The baseline thruster for GP-B was selected based on the minimum control authority plot. Although the focus of this article is on thruster systems, the techniques developed are applicable in general to any system of actuators of any type, for example, the primary flight control effectors of highly maneuverable aircraft.

I. Introduction

THE designer of a thruster system for the attitude and translation control of a spacecraft is faced with many options: how should the thrusters be distributed on the spacecraft, which way should they point, what type of thrusters should be used, how big should they be, what is the best way to control them, and taking redundancy into account, what is the optimum number of thrusters? Above all else, the designer must make sure that the thruster system has enough control authority to overcome all possible disturbance forces. This is guaranteed if the worst case output or minimum control authority is greater than the worst case or largest disturbance force acting on the spacecraft. Various techniques to calculate the minimum control authority of a thruster system are derived in this article. A tool called the minimum control authority plot is also developed, which helps the designer graphically evaluate the various design options. It gives the minimum magnitude moment-generating capability for a specified magnitude of force.

A. Minimum Control Authority Plot

As a simple example of a minimum control authority plot consider the thruster system depicted in Fig. 1. Four thrusters produce forces in the $F_x = F_y$ plane and two additional thrusters (not shown) generate a pure moment (torque) F_M, which is plotted along the vertical axis. If the propellant for the six thrusters is supplied by a single reservoir with a limited flow rate, then the pyramid in Fig. 1 depicts the maximum force and moment that can be generated by the system in any direction. We call this the "control authority" of the system. It is the boundary of the controllable region or the locus of points in the force output space where a limit on the thrusters is first reached. The thrusters can generate higher forces in some directions than in others. The corners at the base of the pyramid correspond to the force that can be generated if all of the propellant is directed out through just one of the force thrusters. The top of the pyramid is the maximum moment achieved if all of the flow is directed out through the two moment-generating thrusters. The heavy line on the face of the pyramid represents the force-vs-moment generating capability in the weakest direction. We call the plot of this line (Fig. 2) the minimum control authority plot. For a given force magnitude $\|F_F\|_2$, this plot gives the minimum or worst case moment magnitude $\|F_M\|_2$ that can be generated by the thruster system.

By plotting the maximum expected force and moment disturbance the thruster system must overcome on the same plot, the thruster system can be evaluated relative to what it has to do. If every point of the minimum control authority plot is above the maximum force and moment disturbances, as in Fig. 2, then the thruster system is strong enough to overcome any disturbance even in its weakest output direction.

B. Outline of Article

The minimum control authority plot is given a precise mathematical definition in the next section. To illustrate the concept and demonstrate its usefulness, the minimum control authority plot is then applied to the design of the thruster system for the Gravity Probe B (GP-B)[1,2] spacecraft. Next we go into the details of calculating the minimum control authority. First we discuss how to control the thrusters to achieve a desired output force on the spacecraft while simultaneously minimizing either flow rate, power, or peak thruster force. This is followed by detailed descriptions on how to calculate the minimum control authority for various combinations of thruster controllers and thruster limits. Given these techniques, Sec. VI describes how to generate the minimum control authority plot.

The techniques developed in this aritcle to calculate the minumum control authority plot are completely general and are not restricted just to thruster systems. The techniques can be applied to any parallel configuration of actuators, whether it is a system of reaction wheels, control moment gyros, hydraulic actuators, pneumatic actuators, or parallel robotic manipulators like a robot hand or Stewart platform.[3] All of the statements in this article with the term *thruster* therefore are equally valid if thruster is replaced with the term *actuator*. For example, the techniques could be applied to the next generation of highly maneuverable airplanes that are projected to have as many as 20 primary flight control actuators.[4,5]

The concept of using the minimum control authority to evaluate various thruster configurations for GP-B was originally proposed by Chen.[6]

II. Problem Definition

The simple example of Fig. 1 has only six thrusters. For this example the minimum control authority plot can easily be determined graphically. In general, however, a spacecraft will have many thrusters generating both forces and moments with a total of six-degrees-of-freedom control. The control authority in this case is not a simple pyramid but is some unimaginable surface in six-dimensional space. Because it is impossible to draw this surface,

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*Chief Designer.

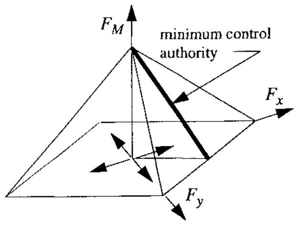

Fig. 1 Control authority of six thrusters with a limit on flow rate.

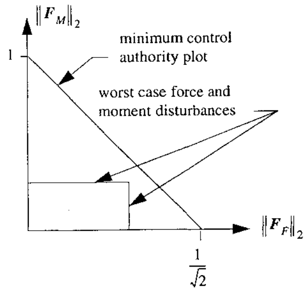

Fig. 2 Minimum control authority plot for system of six thrusters. Flow rate is limited to peak value of 1. Vertical and horizontal lines correspond to worst case applied force and moment loads (disturbances) on system, respectively.

a mathematical approach is needed to calculate the minimum control authority plot for the general six-degrees-of-freedom case. The size and shape of the minimum control authority surface depends on three things: 1) *thruster configuration*, the number of thrusters and their physical placement on the spacecraft; 2) *thruster control*, the way in which the individual thrusters are controlled to obtain a desired output force; and 3) *thruster limit*, the physical mechanism that limits the forces that can be generated by the thruster system. An intermediate step in calculating the minimum control authority plot is to first find the minimum control authority, which is the point on the minimum control authority surface closest to the origin. Finding the minimum control authority involves a quadratic minimization subject to the constraints of a fixed configuration, control, and thruster limit. The goal of the next six subsections is to give a precise definition of the minimum control authority plot. We start by defining thruster configuration and then thruster limit, thruster control, control authority, minimum control authority, and finally the minimum control authority plot.

A. Thruster Configuration

A typical spacecraft, such as the one for GP-B (Sec. III), may have 18 thrusters providing moments $\boldsymbol{F}_M$ for attitude control around three axes and forces $\boldsymbol{F}_F$ for translation control along three axes. The net force and moment exerted on the spacecraft by the combined outputs of the individual thrusters can be expressed as

$$\boldsymbol{F} = A\boldsymbol{T} \tag{1}$$

where $\boldsymbol{F}$ is a 6×1 generalized force vector composed of the three components of the force, $\boldsymbol{F}_F$, and moment, $\boldsymbol{F}_M$, vectors respectively. If n is the number of thrusters, then $\boldsymbol{T}$ is an $n \times 1$ vector of the magnitudes of the force exerted by the individual thrusters and A is the $6 \times n$ configuration matrix that defines the positions and orientations of each thruster. For example, consider the thruster configuration depicted in Fig. 3, consisting of three thrusters radiating from a single point (center of mass). Equation (1) for this case is

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_F \\ \boldsymbol{F}_M \end{bmatrix} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ 0 & 0.87 & -0.87 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \tag{2}$$

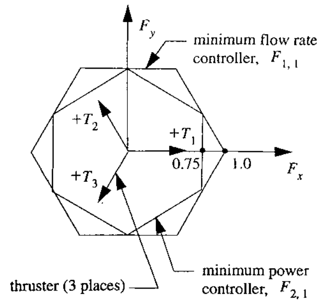

Fig. 3 Control authorities of system of three, two-sided thrusters using either minimum power or minimum flow rate controller. Only positive outputs, $+T_1$, $+T_2$, and $+T_3$, of two-sided thrusters are shown. Flow rate is limited to 1 in both cases.

The number of thrusters n is typically greater than the number of degrees of freedom. This means that the $6 \times n$ configuration matrix A is "fat" and the set of simultaneous linear equations (1) are underdetermined.

B. Thruster Limit

Any actuation system has some physical mechanism that limits its maximum output force generating capability. A thruster or hydraulic system, for example, may be limited by available flow rate, power, or pressure. An electrical system may have a limited current, voltage, or power available to drive the motors. These various physical limitations can be defined in terms of vector norms.[7] These norms are the mathematical basis for the techniques derived later in this article to calculate the minimum control authority of a system of thrusters.

A vector norm measures the size of a vector. For example, the 2-norm $\|\boldsymbol{T}\|_2$ is the length or magnitude of the vector $\boldsymbol{T}$ and corresponds to the square root of the sum of the squares of the elements of $\boldsymbol{T}$:

$$\|\boldsymbol{T}\|_2 \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} T_i^2} \tag{3}$$

If the elements of $\boldsymbol{T}$ are the force outputs of a set of thrusters, then the 2-norm measures the square root of the total power generated by the system. A limit on the net power available to a system of thrusters is characterized by an upper bound on the 2-norm, $\|\boldsymbol{T}_s\|_2 \leq m_2$.

The 1-norm $\|\boldsymbol{T}\|_1$ is the sum of the absolute values of $\boldsymbol{T}$,

$$\|\boldsymbol{T}\|_1 \stackrel{\text{def}}{=} \sum_{i=1}^{n} |T_i| \tag{4}$$

and therefore corresponds to the total flow rate. A system with a finite flow rate or current is characterized by a limit on the 1-norm of the thruster command vector, $\|\boldsymbol{T}_s\|_1 \leq m_1$.

The ∞-norm is the component of $\boldsymbol{T}$ with the largest magnitude,

$$\|\boldsymbol{T}\|_\infty \stackrel{\text{def}}{=} \max_i |T_i| \tag{5}$$

and in our case corresponds to the maximum of the force outputs of the individual thrusters. A system limited by the peak force achievable by the individual thrusters can be characterized by a limit on the ∞-norm, $\|\boldsymbol{T}_s\|_\infty \leq m_\infty$.

C. Thruster Control

A thruster system's main task is to generate desired forces and moments $\boldsymbol{F}$ on the spacecraft. Some type of algorithm is therefore required to decide how to control the individual thrusters to obtain the desired output forces and moments. Controlling the thrusters involves solving Eq. (1), $\boldsymbol{F} = A\boldsymbol{T}$, for $\boldsymbol{T}$ given $\boldsymbol{F}$ and A. Typically Eq. (1) is underdetermined, in which case it can be satisfied by an infinite number of different vectors $\boldsymbol{T}$. A unique solution can be specified by requiring that $\boldsymbol{T}$ satisfies some additional constraints. The most common constraint specifies that the sum of the squares of the elements of $\boldsymbol{T}$ are a minimum. This least squares, min $\|\boldsymbol{T}\|_2$, constraint yields the simple linear thruster controller, $\boldsymbol{T} = A^{\dagger}\boldsymbol{F}$, where $A^{\dagger}$ is the pseudoinverse[8] of A. For reasons that will become apparent shortly, we also study the thruster controllers that minimize either the peak individual thruster force, min $\|\boldsymbol{T}\|_\infty$, or the sum of the absolute values of the outputs of the individual thrusters, min $\|\boldsymbol{T}\|_1$. These three types of controllers yield the desired output force $\boldsymbol{F}$ while minimizing one of the following three physical quantities: 1) min $\|\boldsymbol{T}\|_1$, minimizes fuel flow rate; 2) min $\|\boldsymbol{T}\|_2$, minimizes power; and 3) min $\|\boldsymbol{T}\|_\infty$, minimizes peak individual thruster force.

D. Control Authority

For a given thruster configuration, thruster controller, and thruster limit, we call the surface of maximum achievable output forces in any direction the control authority. For example, this is the pyramid in Fig. 1 or the polygons and circle in Fig. 4. To see how different controller types can affect the control authority, consider the three thrusters in a plane depicted in Fig. 3. The smaller hexagon is the control authority if the minimum power controller, min $\|\boldsymbol{T}\|_2$, is used and the larger hexagon corresponds to the control authority if the minimum flow rate controller, min $\|\boldsymbol{T}\|_1$, is used. The amount of flow rate available to drive the thrusters is limited to the same value in both cases. This illustrates the importance of using the appropriate controller to achieve the greatest possible forcing capability out of the thruster system. The minimum flow rate controller, min $\|\boldsymbol{T}\|_1$, uses the available flow rate as efficiently as possible. For example, to generate a force along the horizontal direction, it only turns on the thruster that points in that direction, $+T_1$. The minimum power controller is not as fuel efficient in this case since it turns the other two thrusters, $-T_2$ and $-T_3$, on equal amounts, wasting fuel in the positive and negative vertical directions. Consequently, the minimum power controller can only generate 75% of the force in the horizontal direction compared to the minimum-flow-rate controller.

The same principle applies to systems limited by power or peak individual thruster force. If a system has a limit on the power available to drive the thrusters, then the greatest possible output force can be achieved with the controller that minimizes power, min $\|\boldsymbol{T}\|_2$, for a given output force. Similarly the greatest output forces can be achieved if the minimum peak individual thruster force controller, min $\|\boldsymbol{T}\|_\infty$, is used for systems limited by the peak force that can be generated by the individual thrusters. The selection logic should be appropriate to the actual physical constraint.

The control authorities for the simple planar, two-degrees-of-freedom examples (Figs. 3 and 4) in this subsection can be depicted as simple polygons or circles. This is not true in general for a thruster system with six-degrees-of-freedom control. Here the control authority is a surface in six-dimensional space and can only be defined in mathematical terms. In general, the control authority $\boldsymbol{F}_{s,p}$ is defined as the set of forces,

$$\boldsymbol{F}_{s,p} \stackrel{\text{def}}{=} A\boldsymbol{T}_s \tag{6}$$

such that

$$\|\boldsymbol{T}_s\|_p = m_p \tag{7}$$

where

$$\boldsymbol{T}_s \stackrel{\text{def}}{=} \arg\min_{T} \|\boldsymbol{T}\|_s \tag{8}$$

and $s, p = 1, 2, \infty$. The first subscript, s, in $\boldsymbol{F}_{s,p}$ refers to the type of controller used: min $\|\boldsymbol{T}\|_s$, where $s = 1, 2, \infty$. The second subscript, p, refers to the type of thruster limit: $\|\boldsymbol{T}\|_p = m_p$, where $p = 1, 2, \infty$. There are nine possible combinations of s and p. For a given thruster configuration matrix the control authority $\boldsymbol{F}_{s,p}$ is the set of maximum achievable forces and moments in any direction for a given thruster controller and thruster limit. The notation arg min in Eq. (8) means that $\boldsymbol{T}_s$ is the specific $\boldsymbol{T}$ that minimizes the s-norm of $\boldsymbol{T}$.

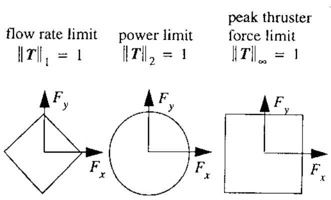

Fig. 4 Control authority of two mutually perpendicular thrusters in a plane for various thruster limits. Each thruster is two sided, being capable of generating forces in both positive and negative directions.

E. Minimum Control Authority

The minimum control authority is the shortest distance to the control authority surface,

$$\min \|\boldsymbol{F}_{s,p}\|_2, \qquad s, p = 1, 2, \infty \tag{9}$$

It corresponds to the weakest output direction of the thruster system. The minimum control authority can be thought of as a minimax problem in the sense that it is the *minimum* over all possible output directions of the *maximum* force that can be generated by the thruster system in any of those directions given a specific thruster controller. Specifying a thruster limit [Sec. II.B] along with the thruster controller [Sec. II.C] automatically specifies a maximum force. The minimax problem therefore reduces to a quadratic minimization, min $\|\boldsymbol{F}_{s,p}\|_2$, subject to the constraints of 1) a specific thruster controller, min $\|\boldsymbol{T}\|_s$, $s = 1, 2, \infty$, and 2) a specific thruster limit, $\|\boldsymbol{T}_s\|_p = m_p$, $p = 1, 2, \infty$.

As a simple example, the minimum control authority of two thrusters with a limit on flow rate, depicted in the leftmost plot of Fig. 4, corresponds to the points on the square closest to the origin. The distance to these points is the minimum control authority.

F. Minimum Control Authority Plot

The minimum control authority, min $\|\boldsymbol{F}_{s,p}\|_2$, is a single number that measures the worst case force and moment-generating capability of a thruster system. The number itself is difficult to interpret, however, since it mixes units of force and moment together. It would be far more useful if this single number could be split into two numbers corresponding to the worst case moment vs worst case force outputs. This is the purpose of the minimum control authority plot, which is a graphical representation of the trade-off between the force- and moment-generating capabilities of the thruster system in the worst case output directions. The minimum control authority plot in Fig. 2, for example, reflects the fact that as more force output is demanded of the thruster system, the amount of flow rate available to generate moments decreases. In the limit, if a force output magnitude of $1/\sqrt{2}$ is commanded, then no flow rate is available to generate moments. In general, given a specific thruster configuration, thruster controller, and thruster limit, the minimum control authority plot indicates the minimum moment magnitude that can be generated by a thruster system for a fixed force magnitude. This worst case moment magnitude is plotted along the vertical axis, and the worst case force magnitude is plotted along the horizontal axis. Calculating the minimum control authority plot involves a quadratic minimization, min $\|\boldsymbol{F}_M\|_2$, given a quadratic constraint, min $\|\boldsymbol{F}_F\|_2 = F_F$, and the following additional constraints: 1) *thruster configuration* $\boldsymbol{F} = A\boldsymbol{T}_s$; 2) *thruster controller* $\boldsymbol{T}_s \stackrel{\text{def}}{=} \arg\min_T \|\boldsymbol{T}\|_s$, $s = 1, 2, \infty$; and 3) *thruster limit* $\|\boldsymbol{T}_s\|_p = m_p$, $p = 1, 2, \infty$. Of course, the decision to plot moment vs force in the minimum control authority plot is a bit arbitrary. The minimum control authority plot could be applied to any pair of decoupled axes of generalized force.

III. Applications to Gravity Probe B

The minimum control authority plot was used to design a thruster system for the GP-B spacecraft, which will carry an experiment to test several aspects of general relativity theory. According to Einstein's theory, the spin axis of a gyroscope placed in orbit around Earth will experience a very small drift of its spin axis relative to inertial space.[9] GP-B will measure this relativistic drift with four gyroscopes electrostatically suspended within a spacecraft. To achieve the experimental goal of measuring these drifts to better than 1% accuracy, the nonrelativistic drift must be less than 0.1 milliarcsec per year.[10] This phenomenally low drift rate is achieved by canceling external disturbances by means of a *drag-free*[11,12] translation

Table 1 Thruster limits and controllers for GP-B

Controller type	Thruster limit $\|T_s\|_1 = v_e w_n$	$\|T_s\|_\infty = v_e \frac{w_n}{n}$	$\|T_s\|_\infty = v_e k w_n$
$\min \|T\|_2$	Line 2	Line 1	Line 3
$\min \|T\|_\infty$	—	—	Line 4

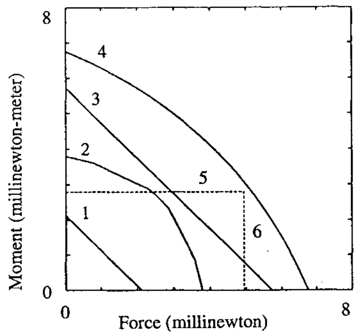

Fig. 5 Minimum control authority plots for GP-B: line 1, differential thrusters; lines 2–4, one-sided thrusters.

control system. Also, a very precise attitude control system maintains inertial pointing of the spacecraft to within 60 milliarcsec.[10] A set of thrusters provides the six-degrees-of-freedom force/moment control authority for both the translation and attitude control systems. The propellant for the thrusters is obtained from the boil-off gas of an on-board liquid helium cryogenic system.[13] Since the average boil-off rate is only $w_n = 5 \times 10^{-6}$ kg/s, the thruster system must be designed very carefully to maximize the minimum control authority. The minumum control authority plot has been applied to evaluate various designs for the GP-B thruster system.[13]

Recall from Sec. I that there are three things that affect the minimum control authority: 1) thruster configuration, 2) thruster controller, and 3) thruster limit. In this example for GP-B we fix the thruster configuration and investigate how the other two factors affect the minimum control authority. The thruster configuration consists of 16 thrusters clustered in groups of 4 at the four vertices of a regular tetrahedron. We investigate the minimum control authority for two types of controllers: 1) minimum power, $\min \|T\|_2$, and 2) minimum thruster output, $\min \|T\|_\infty$. We investigate: three types of thruster limits:

1) Peak individual thruster force limit $\|T_s\|_\infty = v_e(w_n/n)$, where $n = 8$, two-sided, differential thrusters[13]: The differential thrusters divide the total flow rate w_n by n so that the peak flow rate per thruster is w_n/n. The parameter values v_e, w_n, and n are defined in Table 1. This limit corresponds to line 1 in the minimum control authority plot of Fig. 5.

2) Flow rate limit $\|T_s\|_1 = v_e w_n$, where a software limit is imposed on the 16 thrusters so that the total flow rate does not exceed w_n kilograms per second: This type of limit is necessary to maintain uniform cooling for the liquid helium cryogenic system.[13] This limit corresponds to line 2 in Fig. 5.

3) Peak individual thruster force limit $\|T_s\|_\infty = v_e k w_n$, where the peak flow rate per thruster is $k w_n$ kilograms per second: The value k is a function of the thruster throat area and the total flow rate and w_n is determined by the supply pressure.[13] This limit corresponds to lines 3 and 4 in Fig. 5.

Table 1 summarizes the four combinations of thruster limits and controllers we analyze for the given thruster configuration. The minimum control authority plots corresponding to these four cases are displayed in Fig. 5. These plots are based on the parameter values listed in Table 2.

Summary of the minimum control authority plots for GP-B: Figure 5 illustrates how the minimum control authority plot lets the thruster system designer graphically evaluate various design options. For GP-B, for example, we see immediately that one-sided thrusters (lines 2–4) have greater authority than differential thrusters (line 1). Comparing lines 4 and 3, we also see the increased authority realized with the minimum peak individual thruster force controller (line 4) vs the minimum power controller (line 3) for the same thruster limit, $\|T_s\|_\infty = v_e k w_n$. The increased authority must be traded off against the increased computational burden for this type of controller (Sec. IV.B). Lines 5 and 6 in Fig. 5 are the worst case moment and force disturbances expected to act on the GP-B spacecraft.

Table 2 Parameter values for GP-P

Parameter	Value
Exhaust gas velocity	$v_e = 1274$ m/s
Total flow rate	$w_n = 5 \times 10^{-6}$ kg/s
Flow rate ratio	$k = 0.33$
Thruster moment arm	$r = 1$ m
Number of differential thrusters	$n = 8$

The rest of this article goes into the details of controlling the thrusters, calculating the minimum control authority, and generating the minimum control authority plot.

IV. Thruster Control

As outlined in Sec. II.C, controlling the thrusters to obtain a desired force output $\boldsymbol{F}$ given a configuration matrix A involves solving the set of simultaneous linear equations $\boldsymbol{F} = A\boldsymbol{T}_s$ for the thruster command vector $\boldsymbol{T}_s$. In this article we find the three solutions $\boldsymbol{T}_s$ that satisfy $\boldsymbol{F} = A\boldsymbol{T}_s$ and in addition minimize the $s = 1, 2, \infty$ norms of $\boldsymbol{T}$. In mathematical terms we find the following three solutions, $\boldsymbol{T}_s$: given $\boldsymbol{F}$ and A,

$$\boldsymbol{T}_s \stackrel{\text{def}}{=} \arg\min_{\boldsymbol{T}} \|\boldsymbol{T}\|_s \tag{10}$$

such that $\boldsymbol{F} = A\boldsymbol{T}_s$, where $s = 1, 2, \infty$, $\boldsymbol{T} \geq \boldsymbol{0}$. The notation arg min in Eq. (10) means that $\boldsymbol{T}_s$ is that specific $\boldsymbol{T}$ that minimizes the s-norm of $\boldsymbol{T}$ subject to the constraints $\boldsymbol{F} = A\boldsymbol{T}_s$ and $\boldsymbol{T} \geq \boldsymbol{0}$. The inequality constraint is necessary for one-sided thrusters capable of generating forces in only one direction. The techniques used to calculate these three solutions, $s = 1, 2, \infty$, are defined in the next three subsections. For $s = 2$ the solution (thruster controller) is easy: It involves calculating the pseudoinverse of A. The solutions are more difficult if $s = 1$ or $s = \infty$ since they involve solving linear programs.

A. Minimum Power Controller

The solution to $\boldsymbol{F} = A\boldsymbol{T}$ that minimizes power, $\boldsymbol{T}_2 = \min \|\boldsymbol{T}\|_2$, is the simplest to implement since the thruster command vector $\boldsymbol{T}_2$ is simply a linear function of the desired force $\boldsymbol{F}$:

$$\boldsymbol{T}_2 = A^\dagger \boldsymbol{F} \tag{11}$$

The matrix $A^\dagger$ is the pseudoinverse of A and is equal to $A^T(AA^T)^{-1}$. The pseudoinverse can also be conveniently found using the function pinv in MATLAB,[14] which uses a numerically stable algorithm based on the singular value decomposition.[7] For a given thruster configuration A, the pseudoinverse $A^\dagger$ has to be computed only once. Given $A^\dagger$, computing the thruster command vector $\boldsymbol{T}_2$ involves only the simple matrix multiplication in Eq. (11).

This simple controller (11) will not work directly for one-sided thrusters, $\boldsymbol{T}_2 \geq \boldsymbol{0}$. For example, if $\boldsymbol{T}_2 \geq \boldsymbol{0}$ for a desired force $\boldsymbol{F}$, then, from (11), $\boldsymbol{T}_2 \leq \boldsymbol{0}$ for $-\boldsymbol{F}$. By biasing the thruster commands about the middle of their output range, however, the pseudoinverse may be applied without yielding any negative thruster commands.

B. Minimum Flow Rate Controller

The minimum flow rate controller yields the greatest possible output force for a system that has a limit on the total flow rate supplied to the thrusters. The solution $\boldsymbol{T}_1$ that minimizes the flow rate can be found by solving the following linear program[8]:

$$\boldsymbol{T}_1 = \arg\min_{\boldsymbol{T}} \boldsymbol{I}^T \boldsymbol{T} \tag{12}$$

subject to

$$A\boldsymbol{T} \geq \boldsymbol{F}, \qquad \boldsymbol{T} \geq \boldsymbol{0} \tag{13}$$

where $\boldsymbol{I}$ is a vector of ones and $\boldsymbol{0}$ is a vector of zeros of the same length as $\boldsymbol{T}$. In words, Eq. (12) says "minimize the 1-norm of $\boldsymbol{T}$" and Eq. (13) says "subject to the constraints that $\boldsymbol{T}$ generates at least the desired force, $A\boldsymbol{T} \geq \boldsymbol{F}$, using only one-sided thrusters, $\boldsymbol{T} \geq \boldsymbol{0}$."

Linear programs like (12) and (13) can easily be solved using the Simplex method.[8] Many computer programs that implement the Simplex method are available. The Simplex method converges very quickly and therefore can be used for real-time control for many applications. We used the Simplex method, for example, to solve for $\boldsymbol{T}_1$ in the GP-B thruster system. For a thruster system consisting of 18 individual thrusters we were able to solve for 1000 completely random generalized force commands in 80 s. This corresponds to an update rate of 12.5 Hz. The calculations were done on a Sun-3 computer rated at 12 million instructions per second (mips). The program LSSOL[15] was used, which implements the Simplex method in Fortran. Although 12.5 Hz is fast enough for the GP-B translation and attitude control systems, it is not nearly as fast as the simple matrix multiplication of the minimum power controller (11). This controller runs about 100 times faster on the same computer.

C. Minimum Peak Individual Thruster Force Controller

A system that is limited by the peak forces attainable by the individual thrusters produces the largest possible force if the minimum peak individual thruster force controller, min $\|\boldsymbol{T}_s\|_\infty$, is used. This controller is the solution to the following linear program:

$$\boldsymbol{T}_\infty = \arg\min \begin{bmatrix} \boldsymbol{0}^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ z \end{bmatrix} \tag{14}$$

subject to

$$\begin{bmatrix} -I & \boldsymbol{I} \\ A & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ z \end{bmatrix} \geq \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{F} \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} \boldsymbol{T} \\ z \end{bmatrix} \geq \boldsymbol{0} \tag{16}$$

where I is the identity matrix, $\boldsymbol{I}$ is a vector of ones, $\boldsymbol{0}$ is a vector of zeros, and z is a scalar. Another way of stating Eqs. (14–16) is

$$\min z \tag{17}$$

subject to

$$\boldsymbol{T} \leq \boldsymbol{z} \tag{18}$$

$$A\boldsymbol{T} \geq \boldsymbol{F} \tag{19}$$

$$z,\ \boldsymbol{T} \geq \boldsymbol{0} \tag{20}$$

where $\boldsymbol{z}$ is a vector of z the same length as $\boldsymbol{T}$. This linear program finds the solution to $\boldsymbol{F} = A\boldsymbol{T}$ with the smallest ∞-norm on $\boldsymbol{T}$.

V. Calculating the Minimum Control Authority

In this section we present in detail the techniques used to calculate the minimum control authority of a system of thrusters. Since the minimum power solution is the easiest to compute, it is commonly used even if the thruster system is not limited by available power, although this does not yield the greatest control authority. Systems that are limited by thruster output or flow rate, for example, often use the minimum power controller. For the minimum power controller, therefore, we show how to compute the minimum control authority for all three cases: 1) flow rate limited, 2) power limited, and ∞) peak individual thruster force limited. The other two controller types, minimum flow rate, min $\|\boldsymbol{T}\|_2$, and minimum peak individual thruster force, min $\|\boldsymbol{T}\|_\infty$, are so computationally intensive compared to the minimum power controller that they are only used if the increased minimum control authority is essential. For example, the minimum flow rate controller would only be used if the system output was limited by flow rate. Similarly, the minimum thruster force controller would only be used if the output of the system was limited by the peak-force-generating capability of the individual thrusters. The specific combinations of controllers and thruster limits that we analyze in this section are summarized in Table 3.

A. Minimum Power Controller

Not only is the minimum power controller the easiest to compute, it also makes it easy to compute the minimum control authority. Recall from Eqs. (6) and (7) that the control authority is the set of forces $\boldsymbol{F}_{s,p} = A\boldsymbol{T}_s$ satisfying some p-norm on $\boldsymbol{T}_s$, $\|\boldsymbol{T}_s\|_p = m_p$. From Eq. (11) we know that the minimum power solution $\boldsymbol{T}_2$ can be written as a function of $\boldsymbol{F}$, $\boldsymbol{T}_2 = A^\dagger \boldsymbol{F}$. Therefore the minimum control authority is

$$\min \|\boldsymbol{F}_{2,p}\|_2 \stackrel{\text{def}}{=} \min_{\boldsymbol{F}} \|\boldsymbol{F}\|_2 \tag{21}$$

such that

$$\|A^\dagger \boldsymbol{F}\|_p = m_p, \qquad p = 1, 2, \infty \tag{22}$$

Equations (21) and (22) show that if the minimum power controller is used, then the minimum control authority depends only on the input/output properties of the pseudoinverse matrix $A^\dagger$. The input to $A^\dagger$ is $\boldsymbol{F}$ and the output is $A^\dagger\boldsymbol{F}$. Equations (21) and (22) state that the minimum control authority corresponds to the force with the smallest 2-norm that results in an output with a p-norm of $\|A^\dagger\boldsymbol{F}\|_p = m_p$. In other words, if the size of the input is measured by the 2-norm $\|\boldsymbol{F}\|_2$ and the size of the output is measured by the p-norm $\|A^\dagger\boldsymbol{F}\|_p$, then the minimum control authority corresponds to the force that makes the ratio of $\|A^\dagger\boldsymbol{F}\|_p$ to $\|\boldsymbol{F}\|_2$ as big as possible. We call this ratio $A^\dagger_{2,p}$:

$$A^\dagger_{2,p} \stackrel{\text{def}}{=} \max_{\boldsymbol{F}\neq\boldsymbol{0}} \frac{\|A^\dagger\boldsymbol{F}\|_p}{\|\boldsymbol{F}\|_2} \tag{23}$$

The scalar $A^\dagger_{2,p}$ is a matrix norm and corresponds to the peak gain of the matrix $A^\dagger$ when the size of the input is measured by the 2-norm and the size of the output is measured by the p-norm.[7] If a thruster system is operating at its thruster limit, then the numerator in Eq. (23) is fixed, $\|A^\dagger\boldsymbol{F}\|_p = m_p$. The peak gain $A^\dagger_{2,p}$ occurs if the denominator in Eq. (23) is a minimum, which for our case is the minimum control authority $\|\boldsymbol{F}\|_2 = \min \|\boldsymbol{F}_{2,p}\|_2$. Substituting in for the numerator and denominator in Eq. (23) yields

$$A^\dagger_{2,p} = \frac{m_p}{\min \|\boldsymbol{F}_{2,p}\|_2}$$

so the minimum control authority is

$$\min \|\boldsymbol{F}_{2,p}\|_2 = \frac{m_p}{A^\dagger_{2,p}}, \qquad p = 1, 2, \infty \tag{24}$$

In order to calculate the minimum control authority when the minimum power controller is used, therefore, we must first find the matrix norm $A^\dagger_{2,p}$. As we show in the following three subsections, finding $A^\dagger_{2,p}$ for the three cases when $p = 1, 2, \infty$ is easy.

Minimum Power Controller, Power Limited, min $\|\boldsymbol{F}_{2,2}\|_2$

In this subsection we show how to calculate the minimum control authority of a system of thrusters if the minimum power controller $\boldsymbol{T}_2$ is used and if the thruster system has a limit on the total power available, $\|\boldsymbol{T}_2\|_2 = m_2$. From Eq. (24) we know that we must first find $A^\dagger_{2,2}$. It is a well-known fact that the peak gain of the matrix

Table 3 Five combinations of limits and controllers

	Thruster limit		
Controller type	Flow rate, min $\|\boldsymbol{T}\|_1$	Power, min $\|\boldsymbol{T}\|_2$	Peak force, min $\|\boldsymbol{T}\|_\infty$
min $\boldsymbol{T}_1$	min $\|\boldsymbol{F}_{1,1}\|_2$		
min $\boldsymbol{T}_2$	min $\|\boldsymbol{F}_{2,1}\|_2$	min $\|\boldsymbol{F}_{2,2}\|_2$	min $\|\boldsymbol{F}_{2,\infty}\|_2$
min $\boldsymbol{T}_\infty$	—	—	min $\|\boldsymbol{F}_{\infty,\infty}\|_2$

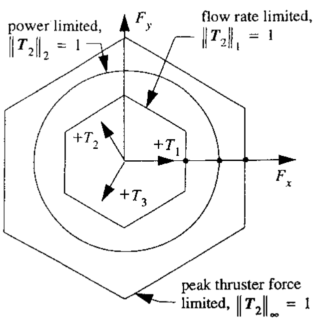

Fig. 6 Control authorities of system of three, two-sided thrusters subject to limitation on flow rate, power, or peak individual thruster force. Only positive outputs, $+T_1$, $+T_2$, and $+T_3$, of two-sided thrusters are shown. Minimum power controller used in all three cases.

$A^\dagger$ when the sizes of both the input and output are measured by the 2-norm is the maximum singular value $\sigma_1(A^\dagger)$ of $A^\dagger$.[8] Therefore, the matrix norm is $A^\dagger_{2,p} = \sigma_1(A^\dagger)$. From Eq. (24) the minimum control authority is

$$\min \|F_{2,2}\|_2 = \frac{m_2}{\sigma_1(A^\dagger)} \tag{25}$$

The maximum singular value $\sigma_1(A^\dagger)$ can be found from the singular-value decomposition $[U, S, V] = \text{svd}(A^\dagger)$, where U, S, and V are matrices such that $A^\dagger = USV^T$. The maximum singular value is the first element in S. The direction of the minimum control authority corresponds to the first column in V, V_1, which has unit magnitude; therefore, $\arg\min \|F_{2,2}\|_2 = \pm \min \|F_{2,2}\|_2 V_1$. The singular-value decomposition can be calculated using various numerical computation software packages like MATLAB.[14]

Example. As a simple example consider three thrusters 120° apart in a plane, as depicted in Fig. 6. The thruster configuration matrix A and the pseudoinverse matrix $A^\dagger$ are, respectively,

$$A = \begin{bmatrix} 1 & -0.5 & -0.5 \\ 0 & 0.87 & -0.87 \end{bmatrix}, \qquad A^\dagger = \begin{bmatrix} 0.67 & 0 \\ -0.33 & 0.58 \\ -0.33 & -0.58 \end{bmatrix} \tag{26}$$

If the power available to the thruster system is limited to $\|T_2\|_2 \le 1$, then the control authority is depicted by the circle in Fig. 6. The maximum singular value of $A^\dagger$ is $\sigma_1(A^\dagger) = \sqrt{\frac{2}{3}}$; the minimum control authority from Eq. (25) is $\min \|F_{2,2}\|_2 = \sqrt{\frac{3}{2}}$. Since all of the singular values of $A^\dagger$ are equal to $\sqrt{\frac{2}{3}}$, the minimum control authority is the same in all directions, as illustrated by the circle in Fig. 6.

Minimum Power Controller, Peak Individual Thruster Force Limited, $\min \|F_{2,\infty}\|_2$

Here we show how to compute the minimum control authority of a thruster system whose output force is limited by the peak force that can be generated by the individual thrusters. Even though the minimum power controller does not result in the greatest possible minimum control authority for this type of thruster limit (see, e.g., Fig. 7), it is still used very frequently since it requires the least amount of computations. From Eq. (24), the minimum control authority for this case is

$$\min \|F_{2,\infty}\|_2 = \frac{m_\infty}{A^\dagger_{2,\infty}} \tag{27}$$

The matrix norm $A^\dagger_{2,\infty}$ is one of the easiest to compute. It is equal to the maximum 2-norm of the rows of $A^\dagger$. If a_i^T are the rows of $A^\dagger$,

$$A^\dagger = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix} \tag{28}$$

and if a_m^T is the row with the largest 2-norm, $a_m^T \overset{\text{def}}{=} \arg\max_i \|a_i^T\|_2$, then $A^\dagger_{2,\infty} = \|a_m\|_2$. See Ref. 13 for a proof of this fact. From Eq. (27), therefore, the minimum control authority is

$$\min \|F_{2,\infty}\|_2 = \frac{m_\infty}{\|a_m\|_2} \tag{29}$$

The minimum control authority is in the same direction as the row with the largest 2-norm:

$$\arg\min \|F_{2,\infty}\|_2 = \pm \min \|F_{2,\infty}\|_2 \frac{a_m}{\|a_m\|_2} \tag{30}$$

Example. Consider three thrusters in a plane again (Fig. 6). If the individual thrusters have a limit on their peak force normalized to a value of 1, $\|T_2\|_\infty \le 1$, and if the minimum power controller is used, then the control authority is the large hexagon in Fig. 6. The pseudoinverse $A^\dagger$ is again given by Eq. (26). The minimum control authority depends on the row with the largest 2-norm in $A^\dagger$, which in this case is $\|a_m\|_2 = \frac{2}{3}$. From Eq. (29) the minimum control authority is $\min \|F_{2,\infty}\|_2 = \frac{3}{2}$. From Eq. (30) the directions corresponding to the minimum control authority are

$$\arg\min \|F_{2,\infty}\|_2 = \begin{cases} \pm[1.5 \quad 0.0]^T \\ \pm[-0.74 \quad 1.3]^T \\ \pm[-0.74 \quad -1.3]^T \end{cases}$$

These are the points on the larger hexagon in Fig. 6 that are closest to the origin.

Minimum Power Controller, Flow Rate Limited, $\min \|F_{2,1}\|_2$

If the flow rate to the thruster system is limited but the minimum power controller is used, then calculating the minimum control authority involves finding $A^\dagger_{2,1}$ from Eq. (24). The technique for finding $A^\dagger_{2,1}$ is similar to the technique described above for finding $A^\dagger_{2,\infty}$, with the additional complication that we must first define the matrix Q.

Definition 1. If n is the number of thrusters, let $Q(n)$ be the $2^n \times n$ matrix formed by counting in binary to 2^n with the 0 replaced by $+1$ and the 1 replaced by -1. For example, if $n = 2$, then,

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \underset{\text{binary}}{\longrightarrow} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \underset{\substack{0\to 1 \\ 1\to -1}}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \overset{\text{def}}{=} Q \tag{31}$$

The absolute value of a negative number is its negative, $|T_i| = -T_i$ if $T_i \le 0$, but it is the number itself if it is positive, $|T_i| = T_i$ if $T_i \ge 0$. The matrix Q therefore accounts for all of the ways that the outputs from the thrusters can be added and subtracted to yield the sum of the absolute values, $\|T\|_1$.

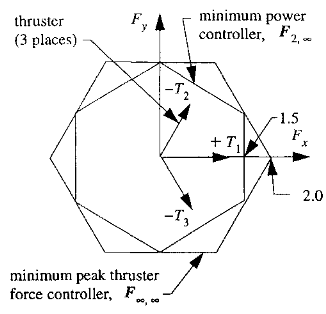

Fig. 7 Control authorities of system of three two-sided thrusters using either minimum power or minimum peak individual thruster force controller. Maximum force of individual thrusters limited to 1 N in both cases. Only positive output, $+T_1$, and negative outputs, $-T_2$ and $-T_3$, of two-sided thrusters are shown.

The matrix norm $A^{\dagger}_{2,1}$ is equal to the maximum 2-norm of the rows of $Q(n-1)A^{\dagger}$. More rigorously, if $nr = 2^{n-1}$ and $q_i^T A^{\dagger}$ are the rows of $QA^{\dagger}$,

$$QA^{\dagger} = \begin{bmatrix} q_1^T A^{\dagger} \\ \vdots \\ q_{nr}^T A^{\dagger} \end{bmatrix} \tag{32}$$

and if $q_m^T A^{\dagger}$ is the row with the largest 2-norm, $q_m^T A^{\dagger} = \arg\max_i \|q_i^T A^{\dagger}\|_2$, then $A^{\dagger}_{2,1} = \|q_m^T A^{\dagger}\|_2$. From Eq. (24) the minimum control authority for this case is

$$\min \|F_{2,1}\|_2 = \frac{m_1}{\|q_m^T A^{\dagger}\|_2} \tag{33}$$

and the direction of the minimum control authority is the same as the vector, $q_m^T A^{\dagger}$,

$$\arg\min \|F_{2,1}\|_2 = \pm \min \|F_{2,1}\|_2 \frac{q_m^T A^{\dagger}}{\|q_m^T A^{\dagger}\|_2} \tag{34}$$

For a proof of Eqs. (33) and (34), see Ref. 13.

Example. Consider the same configuration of three thrusters as in the last two examples (Fig. 6). This time the system output is limited by a maximum flow rate of 1 being available to the thrusters, $\|T_2\|_1 = 1$. To calculate the minimum control authority, we must first define the matrix Q,

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

The matrix $QA^{\dagger}$ is therefore

$$QA^{\dagger} = \begin{bmatrix} 0 & 0 \\ 0.67 & 1.15 \\ 0.67 & -1.15 \\ 1.33 & 0 \end{bmatrix}$$

The last three rows of $QA^{\dagger}$ have a 2-norm of $\|q_m^T A^{\dagger}\|_2 = \frac{4}{3}$. Therefore, from Eq. (33) the minimum control authority is $\min \|F_{2,1}\|_2 = \frac{3}{4}$, and the directions corresponding to the minimum control authority from Eq. (34) are

$$\arg\min \|F_{2,1}\|_2 = \begin{cases} \pm[0.38 \quad 0.65]^T \\ \pm[0.38 \quad -0.65]^T \\ \pm[0.75 \quad 0\ ]^T \end{cases}$$

The control authority for this case is the smaller hexagon in Fig. 6. The minimum control authority corresponds to the points on the hexagon that are closest to the origin.

B. Minimum Flow Rate Controller, Flow Rate Limited, $\min \|F_{1,1}\|_2$

Calculating the minimum flow rate controller requires solving a linear program. To find the minimum control authority of a system using this type of controller, a nonlinear program must be solved. In this section we show how to calculate the minimum control authority of a system that has a limit on flow rate and uses the minimum flow rate controller. In other words, we find $\min \|F_{1,1}\|_2$, which is the shortest distance to the control authority surface, $F_{1,1}$. The control authority $F_{1,1}$ is the set of output forces attainable by the system if the minimum flow rate controller T_1 is used and if the system is operating at its maximum flow rate limit,

$$\|T_1\|_1 = I^T T_1 = m_1 \tag{35}$$

From the previous subsection we know that T_1 satisfies the linear program defined in Eqs. (12) and (13). The dual[8] to this linear program is

$$\max y^T F \tag{36}$$

such that

$$A^T y \le I \qquad \text{and} \qquad y \ge 0 \tag{37}$$

The duality theorem of linear programming states that for an optimal solution the minimum of the primal problem (12) and the maximum of the dual problem (36) have the same value,[8] $y^T F = I^T T_1$. From Eq. (35) this becomes

$$y^T F = m_1 \tag{38}$$

The beauty of the duality theorem is that it allows us to replace the requirement that T_1 solves a linear program with the simple scalar constraint of Eq. (38). Combining Eqs. (38), (13), (37), and (35) results in the following formulation for the minimum control authority:

$$\min \|F_{1,1}\|_2 = \min_F \|F\|_2 \tag{39}$$

such that

$$y^T F = m_1 \qquad AT \ge F \qquad A^T y \le I$$
$$I^T T = m_1 \qquad T, y \ge 0$$

The unfortunate aspect of this minimization problem is that the constraint $y^T F = m_1$ involves the product of two variables, so it is nonlinear. Finding the minimum control authority therefore requires minimizing a quadratic objective function (39) subject to a nonlinear constraint. Nonlinear programs of this type are readily solved using the program MINOS.[16]

Example. Consider three thrusters 120 deg apart in a plane, as in Fig. 3. The two hexagons in the figure correspond to the control authority of the system if the total flow rate is limited to 1, $\|T_s\|_1 \le 1$. The larger hexagon is the control authority if the minimum flow rate controller is used, $s = 1$, and the smaller hexagon corresponds to the minimum power controller, $s = 2$. The minimum flow rate controller is more efficient. For example, if a unit force along the positive x axis is desired, the minimum flow rate controller turns on just the thruster pointing in that direction, $T_1 = [1\ 0\ 0]^T$. The minimum power controller also turns on the other two thrusters, $T_1 = [0.67\ -0.33\ -0.33]^T$, using more fuel. Figure 3 graphically illustrates the increased force that can be achieved by using the appropriate controller for the given thruster limit.

C. Minimum Peak Individual Thruster Force Controller, Peak Individual Thruster Force Limited, $\min \|F_{\infty,\infty}\|_2$

This case is analogous to the minimum flow rate controller, where again the minimum control authority is the solution to a nonlinear program. The minimum control authority of a thruster system limited by the peak forces attainable by the individual thrusters and that uses the minimum peak individual thruster force controller is

$$\min \|F_{\infty,\infty}\|_2 = \min_F \|F\|_2 \tag{40}$$

such that

$$\|T_\infty\|_\infty = m_\infty \tag{41}$$

where T_∞ is the minimum peak individual thruster force controller, $T_\infty = \min \|T\|_\infty$, and therefore satisfies the linear program in Eqs. (14–16). The dual to this linear program is

$$\max\, [y]^T \begin{bmatrix} 0 \\ F \end{bmatrix}$$

Subject to

$$\begin{bmatrix} -I & A^T \\ I^T & 0^T \end{bmatrix} y \le \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{42}$$

$$y \ge 0 \tag{43}$$

From the duality theorem of linear programming (Sec. V.B),

$$[0^T \quad 1]\begin{bmatrix} T \\ z \end{bmatrix} = y^T\begin{bmatrix} 0 \\ F \end{bmatrix}. \tag{44}$$

Combining Eqs. (40), (44), (15), (42), (41), (16), and (43), the minimum control authority is

$$\min \|F_{\infty,\infty}\|_2 = \min \|F\|_2 \tag{45}$$

subject to

$$[0^T \quad 1]\begin{bmatrix} T \\ z \end{bmatrix} = [y]^T\begin{bmatrix} 0 \\ F \end{bmatrix}$$

$$\begin{bmatrix} -I & I \\ A & 0 \end{bmatrix}\begin{bmatrix} T \\ z \end{bmatrix} \geq \begin{bmatrix} 0 \\ F \end{bmatrix}$$

$$\begin{bmatrix} -I & A^T \\ I^T & 0^T \end{bmatrix}[y] \leq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$z = m_\infty$$

$$T,\ y,\ z \geq 0$$

The constraint in Eq. (44) is nonlinear, which means that finding the minimum control authority, $\min \|F_{\infty,\infty}\|_2$, requires minimizing a quadratic objective function, Eq. (45), with a nonlinear constraint. Again this type of problem is readily solved using the program MINOS.[16]

Example. Once again consider three thrusters in a plane (Fig. 7). This time the two hexagons correspond to the control authority of the thruster system if the individual thrusters are limited by their output force, $\|T\|_\infty \leq 1$. The larger hexagon shows the increased control authority obtained by minimizing thruster output ($\min \|T\|_\infty$ controller). The smaller hexagon corresponds to the minimum power controller. The minimum control authority vector for the minimum power controller is $\arg\min \|F_{2,\infty}\|_2 = [1.5 \quad 0]^T$. The thruster forces that yield $\arg\min \|F_{2,\infty}\|_2$ and minimize power are $T_2 = A^\dagger(\arg\min \|F_{2,\infty}\|_2)$. Therefore, from Eq. (26), $[T_1 \quad T_2 \quad T_3]^T = [1 \quad -0.5 \quad -0.5]^T$. The three thruster outputs T_2 are shown in Fig. 7. Notice that T_2 satisfies the constraint $\|T_2\|_\infty \leq 1$. The vector $T_\infty = [1 \quad -1 \quad -1]^T$ also satisfies $\|T_\infty\|_\infty \leq 1$ but yields the higher output $F_{\infty,\infty} = AT_\infty = [2 \quad 0]^T$. The minimum peak individual thruster force controller $T_\infty = \min \|T\|_\infty$ does the smarter thing in this case: It turns each thruster on full blast. It uses more power than T_2 but yields a higher output. The minimum control authority for the minimum peak individual thruster force controller is $\min \|F_{\infty,\infty}\|_2 = 1.73$ and can be found by minimizing Eq. (45) in MINOS.

VI. Generating the Minimum Control Authority Plot

As the force generated by a thruster system increases, its ability to generate moments decreases. The minimum control authority plot is a graphical representation of this interdependence. For a given force magnitude, the minimum control authority plot gives the minimum of the maximum moment. This concept could also be applied to any two independent components of the generalized force vector.

The loads or disturbances a thruster system must overcome are often conveniently expressed in terms of forces and moments. By plotting the force and moment load lines on the same plot as the minimum control authority plot, we can tell whether the system is strong enough to overcome the load. If the minimum control authority plot is above the load lines, then we know that the thruster system can handle the loads under worst case conditions (Fig. 2).

Finding the minimum control authority plot involves normalizing the thruster configuration matrix A before applying the techniques described in Sec. V to calculate the minimum control authority. For example, consider the thruster configuration depicted in Fig. 8, consisting of two thrusters each located a distance r from the center

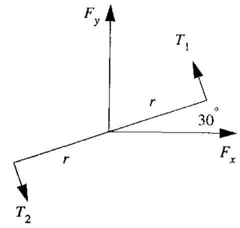

Fig. 8 Planar configuration of two thrusters.

of mass. The thruster configuration matrix for this case is

$$F \overset{\text{def}}{=} \begin{bmatrix} F_F \\ F_M \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} -0.5 & 0.5 \\ 0.87 & -0.87 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ r & r \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \overset{\text{def}}{=} AT \tag{46}$$

where F_F and F_M are 3×1 vectors corresponding to force and moment outputs, respectively. Equation (46) can be normalized by dividing the last three rows of both sides by $\hat{r}$,

$$\hat{F} \overset{\text{def}}{=} \begin{bmatrix} F_F \\ \hat{F}_M \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} -0.5 & 0.5 \\ 0.87 & -0.87 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ r/\hat{r} & r/\hat{r} \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \overset{\text{def}}{=} \hat{A}T \tag{47}$$

The scalar $\hat{r}$ can be any number, for example, the greatest moment arm physically obtainable. The vector $\hat{F}$ is now a normalized generalized force vector. All of the components of $\hat{F}$ have units of force. The matrix $\hat{A}$ is a normalized version of A.

Lam[17] has proposed a different normalization for A. By dividing each row of A by the worst case disturbance force component associated with that row, the minimum control authority is automatically adjusted to the disturbance force and moment distribution. The minimum control authority now gives the safety margin relative to the worst case disturbance force.

Calculating the points along the minimum control authority plot involves computing the minimum control authority of the normalized configuration matrix (47) as the normalizing coefficient $\hat{r}$ is varied between zero and infinity. Values of $\hat{r}$ less than 1 increase the normalized moment arms $r/\hat{r}$ in Eq. (47). The result is to skew the minimum control authority corresponding to the normalized configuration matrix $\hat{A}$ in the direction of pure force outputs which are smaller compared to the moment outputs. For example, in the limit $\hat{r} \to 0$ ($r/\hat{r} \to \infty$), the force output corresponding to $\hat{A}$ is so small compared to the moment output that the minimum control authority is entirely in the direction of force output only. This is the intersection of the minimum control authority plot with the x axis. Similarly, values of $\hat{r} > 1$ skew the minimum control authority plot away from force and toward pure moment outputs. All of the points of the minimum control authority plot from the intersection with the x axis to the intersection with the y axis can be generated as $\hat{r}$ goes from zero to infinity.

The technique for computing the minimum control authority plot involves the following steps:

1. Normalize the configuration matrix A by $\hat{r}$.
2. Compute the minimum control authority vector $\arg\min \|F_{s,p}\|_2$ using any of the techniques in Sec. V depending on the type of thruster limit and thruster controller.
3. Compute the force magnitude $\|F_F\|_2$ and the normalized moment magnitude $\|\hat{F}_M\|_2$ from $\arg\min \|F_{s,p}\|_2$. Note, $\arg\min \|F_{s,p}\|_2 = [F_F^T \quad \hat{F}_M^T]^T$.
4. Find the true moment magnitude $\|F_m\|_2 = \hat{r}\|\hat{F}_M\|_2$.
5. Plot the point $\|F_M\|_2$ vs $\|F_F\|_2$.
6. Repeat the procedure for various $\hat{r}$ between zero and infinity to fill in the minimum control authority plot.

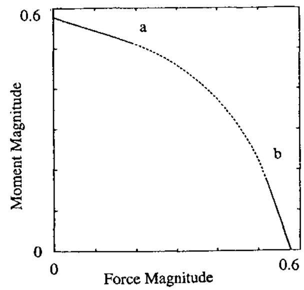

Fig. 9 Discontinuity in minimum control authority plot.

If this procedure is followed, the minimum control authority plot will generally have a break in the middle, as depicted by the dotted line in Fig. 9. The discontinuity occurs at $r/\hat{r} = 1$. It is due to the fact that if $r/\hat{r} < 1$, then the minimum control authority is mostly in the moment direction, but for $r/\hat{r} > 1$ it favors the force direction. Since the two points at the discontinuity, points a and b in Fig. 9, both correspond to $r/\hat{r} = 1$, they consequently have the same minimum control authority, min $\|F_{s,p}\|_2$. The discontinuity can therefore be completed, or "filled in," with the section of a circle corresponding to a constant minimum control authority:

$$\|F_M\|_2 = \sqrt{(\min \|F_{s,p}\|_2)^2 - \|F_F\|_2^2} \tag{48}$$

Equation (48) is the dotted line in Fig. 9. It smoothly connects the discontinuity between the points a and b. Minimum control authority plots for the GP-B thruster system are shown in Fig. 5.

VII. Conclusion

The minimum control authority plot can tell you if a thruster system has sufficient authority to counter disturbances. If the minimum control authority plot is above the greatest possible disturbance forces/moments acting on the system, then the thrusters are strong enough to overcome these disturbances under worst case conditions. The minimum control authority plot is a design tool. It lets the designer of an actuation system know what his/her options are and gives him/her a tool to evaluate competing designs. One of the main features of the minimum control authority plot is that it allows you to evaluate the redundancy of a thruster system. If the minimum control authority of a system of thrusters under a worst case thruster failure is greater than the worst case disturbance load, then the system is redundant. Although we have focused our attention on thruster systems in this article, the techniques are applicable in general to any system of actuators of any type.

Acknowledgments

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